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Quadratics – Major Investigation

Quadratics can be defined as a polynomial equation of a second degree, which implies that it comprises a minimum of one term that is square. The general form of the quadratic equation is $ax^2 + bx + c = 0$. The word quadratic is derived from the word quad, which means square. Graphically, a quadratic expression describes the path followed by a parabola, and it helps in finding things like the height and time of flight of a rocket.

History

The knowledge of ancient civilizations is based only on what survives today. The earliest known problems that led to quadratic equations are on Babylonian tablets dating from around 2000 BCE. In these problems, the Babylonians were trying to find two numbers, x and y, that satisfy the system {x + y = b and xy = c.

The method of solving quadratic equations has evolved over thousands of years, with significant contributions from ancient civilizations. The Babylonians were proficient in solving quadratic problems, employing algorithms that can be seen as precursors to the modern quadratic formula. They, along with the Egyptians, Greek, Chinese, and Indians used geometric methods to solve these equations. The Greeks, for instance, solved quadratics by geometrically constructing rectangles and squares. The Indian mathematician Brahmagupta provided a partial solution to the quadratic equation in his 7th century work, which was later expanded on by other mathematicians. The culmination of these efforts is the quadratic formula, which provides the roots of the equation $ax^2 + bx + c = 0$. It is a staple in modern mathematics education.

Proof of Quadratic Formula

1. Consider $ax^2 + bx + c = 0$, where $a \neq 0$.
2. Since $a \neq 0$, the equatoin can be divided by a to get: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$
3. Complete the square.
4. This shows that the original equation is equivalent to: $\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$
5. Since x appears only once in the equation, it can be rearranged to solve for x.
6. Get the squared term on one side of the equation. $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$
7. The right-hand side can be rewritten by putting it over a common denominator. $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$
8. The square root of both sides can be taken.
9. Taking the possibility of positive and negative square roots into account, the equation at this stage is: $x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$
10. Subtracting $\frac{b}{2a}$ from both sides and putting the right-hand side over a common denominator gives:

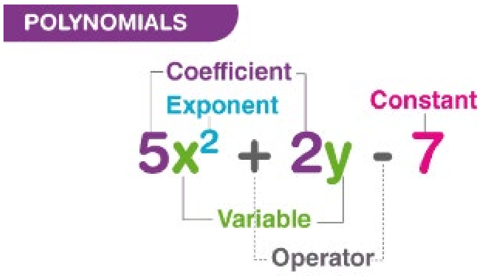
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Real Life Applications

Quadratic equations are used in many career fields and everyday activities, some of which include Astrology, Engineering, Agriculture, Sciences, Military and Sports. They can be used to calculate the areas of an enclosed space, the speed of an object, the profit and loss of a product, or curving a piece of equipment for design.

QUADRATICS MULTIPLYING AND FACTORING

Polynomials



A polynomial is defined as an expression which is composed of variables, constants, and exponents, that are combined using mathematical operations such as addition, subtraction, multiplication and division (no division operation by a variable). A polynomial is made up of 2 or more terms, hence for its name Poly (meaning "many") and Nominal (meaning "terms"). Based on the number of terms present in the expression, it is classified as monomial (1 term), binomial (2 terms), and trinomial (3 terms). Polynomials with more than 3 terms are simply known as polynomials. Some examples of constants, variables and exponents includes:

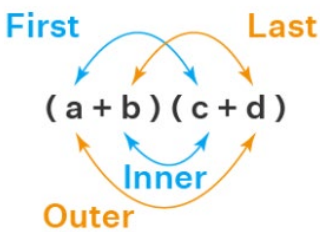
- Constants – 1, 2, 3, 4 etc.
- Variables – g, h, x, y, etc.
- Exponents – 2, 3, 4, 5, etc. (x^2)

The degree of a polynomial is the largest degree of the individual terms. The degree of the term is the sum of the exponents of the variable. E.g. $2x^4y^3$. 4 + 3 is 7 so 7 is the degree of the term. If this term had the largest degree in the polynomial, the degree of the polynomial would be 7. If a term has a negative exponent, it is not a term. E.g. $5x^{-2}y^5$ is not a term due to the negative exponent. If a term consists only of a non 0 number (known as a constant term) its degree is 0. The only term with no degree at all is 0.

Multiplying Binomials

A common method for multiplying binomials is the FOIL method. The word FOIL stands for:

- F – First
- O – Outer
- I – Inner
- L – Last



This method is restricted to binomials only and hence not applicable to all polynomial multiplications. The general form of the FOIL formula is: $(a + b)(c + d) = ac + ad + bc + bd$.

Special Products of Binomials

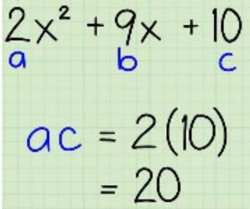
- $(a + b)(a - b) = a^2 - b^2$
- $(a + b)(a + b) = (a + b)^2 = a^2 + 2ab + b^2$
- $(a - b)(a - b) = (a - b)^2 = a^2 - 2ab + b^2$

Factoring Quadratics by Grouping

A quadratic in the form $ax^2 + bx + c$ can be factored into binomial form by using the grouping method.

1. Find the master product by multiplying the a and c terms together. The product of these two terms is referred to as the master product.

- E.g. $2x^2 + 9x + 10$.
- $(a = 2)$ and $(c = 10)$. $(a) * (c) = 20$

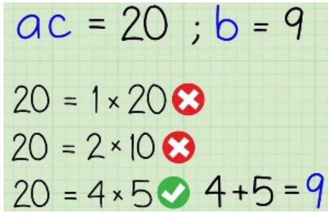


2. Separate the master product into its factor pairs.

- E.g. a factor pair of 20 is (4, 5), as 4 * 5 = 20.

3. Find a factor pair with a sum equal to b, and a product equal to c. If the master product was negative, then the pair of factors must equal the b term when they are subtracted from one another.

- E.g. $2x^2 + 9x + 10$
- (4 + 5 = 9) and (4 * 5 = 20), so this is the correct factor pair.



4. Split the center term and rewrite it, breaking it apart into the factor pair previously identified. Ensure that the proper signs are included. (plus, or minus)

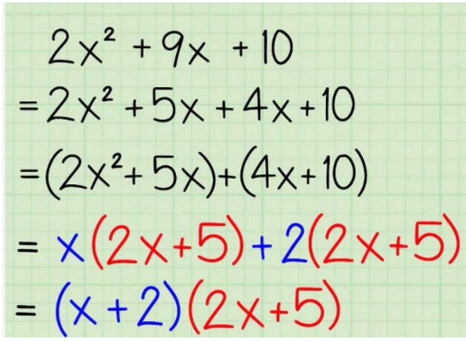
- E.g. $2x^2 + 9x + 10 = 2x^2 + 5x + 4x + 10$.

5. Group the first two terms into a pair and the second two terms into a pair.

- E.g. $2x^2 + 5x + 4x + 10 = (2x^2 + 5x)\ (4x + 10)$

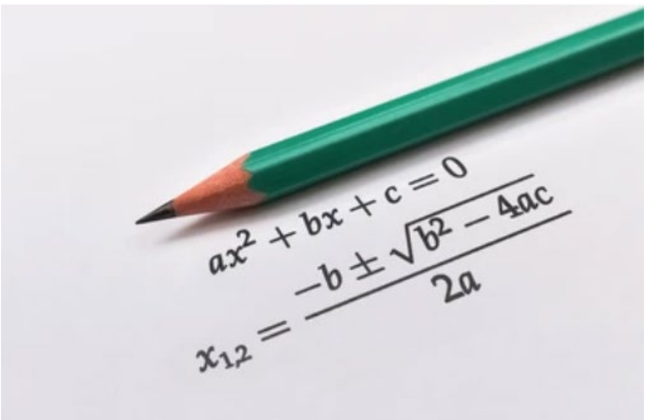
6. Find the least common factors of the pair and factor them out. Rewrite the equation accordingly.

- E.g. $(2x^2 + 5x)\ (4x + 10) = x\ (2x + 5) + 2(2x + 5)$



7. Factor out the shared parentheses. There should be a shared binomial parenthesis between the two halves. Factor this out and place the other terms in another parenthesis. This will be the factored product.

- E.g. $x\ (2x + 5) + 2(2x + 5) = (x + 2)\ (2x + 5)$



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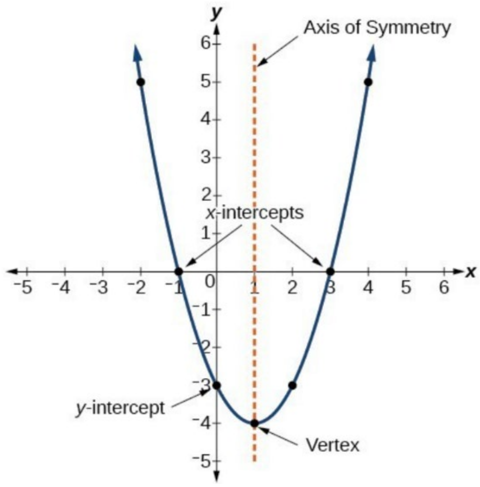
QUADRATICS FUNCTIONS AND EQUATIONS

Parabolas

The graph of a quadratic function is a U – shaped curve called a parabola. An important feature of the graph is that it has an extreme point, called the vertex. If the parabola opens upwards, the vertex represents the lowest point on the graph, or the minimum value of the quadratic function. If the parabola opens downwards, the vertex represents the highest point on the graph, or the maximum value.



Whether it is an upward opening parabola or a downward opening parabola, the vertex is the turning point on the graph. The graph is also symmetric with a vertical line drawn through the vertex, called the axis of symmetry.

The y intercept is the point at which the parabola crosses the y-axis. The x intercepts are the points at which the parabola cross the x-axis. If they exist, the x intercepts represent the zeros, or roots, of the quadratic function; the values of x at which y = 0. A parabola always has a y intercept, but may only have 1, or 0 x intercepts depending on its function.

The general form of a quadratic funtion presents the function in the form:

- $f(x) = ax^2 + bx + c$

where a, b and c are real numbers, and $a \neq 0$. If $a > 0$, the parabola opens upwards. If $a < 0$, the parabola opens downwards. We can use the general form of a parabola to find the equation for the axis of symmetry.

Vertex Form

The Vertex formula of a parabola is f (x) = a (x – h)2 + k, which is used to find the coordinates of the point where the parabola crosses its axis of symmetry. The vertex is the point (h, h). In this form, it is easy to identify the minimum or maximum point of a parabola, which is the vertex. A quadratic equation in standard form can be converted to vertex form throughout numerous steps.

- E.g. an example of an equation in standard form could be $y = -3x^2 - 6x - 9$.
    1. Ensure that the coefficient of x^2 is 1. If the coefficient of x^2 is not 1, we will place the number outside as a common factor. In this case, $y = -3x^2 - 6x - 9 = \underline{-3\,(x^2 + 2x + 3)}$
    2. Identify the coefficient of x
    3. Half the coefficient of x and square the resultant number.
    4. Add and subtract the above number after the x term in the expression.
    (continued on next page)

$$y = -3(x^2 + \boxed{2}\,x + 3)$$

Step 2.

Coefficient of x

$$\left(\frac{2}{2}\right)^2 = 1^2 = 1$$

Step 3.

Step 4.

$$y = -3\left(x^2 + 2x + \underbrace{1 - 1} + 3\right)$$

1 is added and subtracted

$$y = -3\left(\underbrace{x^2 + 2x + 1} - 1 + 3\right)$$

Step 5.

$$y = -3\left((x + 1)^2 - 1 + 3\right)$$

5. Factorise the perfect square trinomial formed by the first 3 terms using the suitable identity. In this case, the formula $x^2 + 2xy + y^2 = (x + y)^2$ can be used. Therefore, $x^2 + 2x + 1 = (x + 1)^2$.
6. Simplify the last two numbers and distribute the outer number. Here, $-1 + 3 = 2$. Therefore, the expression becomes $y = -3(x + 1)^2 - 6$. This is the form of $y = a(x - h)2 + k$, which is the vertex form. Here, the vertex is $(h, k) = (-1, 6)$

Step 6.

$$y = -3\left((x + 1)^2 + 2\right)$$

$$y = -3(x + 1)^2 - 6$$

Zero Product Property

The Zero Product Property simply states that if $ab = 0$, then either $a = 0$ or $b = 0$ (or both). A product of factors is zero if and only if one or more of the factors is zero. This is particularly useful when solving quadratic equations. The zeros of a parabola are the points on the parabola that intersect the line at $y = 0$ (the horizontal x-axis).

- E.g. $x^2 + x - 20 = 0$
- If we factor this equation out, then $x^2 + x - 20 = (x + 5)(x - 4)$
- Therefore, abiding by the zero-product property, either $(x + 5)$ must equal zero or $(x - 4)$ must equal zero (or both).
- $(x = -5)$ OR $(x = 4)$. These are the x intercepts of the parabola.

Wide World of Maths Major Investigation – Self Study on Quadratics

- Vanessa Kong