SOLUTIONS to SAMPLE QUESTIONS

JUNIOR – Years 7 & 8

Solution 1 J

The prime factorisation of 108 is $2^2 \times 3^3$. The five 2's and 3's in these factors must be split up between the three numbers, so one number receives only one of them. Since 2 is too small, this number must be 3. The remaining two 2's and two 3's must be split between the remaining two numbers, with each receiving two primes. Since they are different numbers, this can only be done by one receiving two 2's and the other two 3's. That is, the second number is 2^2 and the third is 3^2. So the three numbers are 3, 4 and 9.

Solution 2 J

The watch loses 96 ÷ 24 = 4 minutes per hour, so advances only 56 minutes every hour. From 8am until the time it shows 4pm it has advanced 14 hours, and so the time taken for this is 14 × 60 ÷ 56 = 15 hours. This means the correct time is 5pm.

Solution 3 J

If T was the total number of jellybeans originally, then $\frac{1}{2}T$ were red, $\frac{1}{3}T$ were blue and $\frac{1}{6}T$ were green. Amanda took $\frac{1}{3} \times \frac{1}{2}T = \frac{1}{6}T$ red ones, and $\frac{1}{2} \times \frac{1}{6}T = \frac{1}{12}T$ green ones. This left $\frac{1}{2}T - \frac{1}{6}T = \frac{1}{3}T$ red ones, of which Cassandra took half, leaving $\frac{1}{6}T$ red. Cassandra also *left behind* $\frac{3}{4} \times \frac{1}{3}T = \frac{1}{4}T$ blue ones, and there were $\frac{1}{12}T$ green remaining. So the total number remaining was $\frac{1}{6}T + \frac{1}{4}T + \frac{1}{12}T = \frac{1}{2}T$. This was 54, so T was 108.

Solution 4 J

Having two children there are four possibilities: the first can be a girl or a boy, and for each of these, the second can be a girl or a boy. We can represent the possibilities by GG, GB, BG and BB, each having equal chances. Knowing that at least one is a girl eliminates one of the cases, which leaves three: GB, BG and GG, each of which occurred with the same probability. So the chances are 1 in 3 $\left(\text{i.e. the probability is } \frac{1}{3}\right)$.

Solution 5 J

Since $2^{\left(2^2\right)} = 2^4 = 16 = 4^2 = \left(2^2\right)^2$, all ways of parenthesising a tower of three 2's give the same result. The towers of four 2's can be categorised according to the bottom-most 2: either it has the form $2^{(T)}$ where T denotes a tower of three 2's (which must give the result 2^{16}), or the bottom-most 2 is inside parentheses containing just two 2's (which gives $\left(2^2\right)^{\left(2^2\right)} = 4^4 = 2^8$) or the bottom-most 2 is in parentheses containing a tower of three 2's (which must give $16^2 = \left(2^4\right)^2 = 2^8$). Thus there are only two numbers which can be written this way: 2^{16} and 2^8.

SOLUTIONS to SAMPLE QUESTIONS

INTERMEDIATE – Years 9 & 10

Solution 1 I

By the definition of "Δ", $(1\,\Delta\,2) = 1 - \frac{1}{2} = \frac{1}{2}$, and $(3\,\Delta\,4) = 1 - \frac{3}{4} = \frac{1}{4}$.

So $(1\,\Delta\,2)\,\Delta\,(3\,\Delta\,4) = \left(\frac{1}{2}\,\Delta\,\frac{1}{4}\right) = 1 - \frac{\frac{1}{2}}{\frac{1}{4}} = 1 - 2 = \text{-}1$

Solution 2 I

Consider pairing the terms in the sequence. $-7 + 14 = -21 + 28 = -35 + 42 = \cdots = 7$. Clearly we need 20 of these pairs to reach a sum of 140, thus we need 2 × 20 or 40 terms.

Solution 3 I

Let a five-digit integer be written $abcde$ where $0 \le a, b, c, d, e \le 9$, and $a \ne 0$.

We have $a \times b \times c \times d \times e = 2000 = 2^4 5^3$. Thus three of the digits have to be 5. The remaining two digits must be single digit factors of 16. There are thus two possibilities, 4 and 4 or 2 and 8. In the first case, three 5s and two 4s may be ordered in $\frac{5!}{3!\,2!} = 10$ ways. In the second case, three 5s, a 2 and an 8 may be ordered in $\frac{5!}{3!\,1!} = 20$ ways. Thus the total is 30.

Solution 4 I

Taking logarithms to the base 2, we find $\log_2(4) + \sin^2(x)\log_2(16) = 6\sin(x)\log_2(2)$, hence $2 + 4\sin^2(x) = 6\sin(x)$. This factorizes to give $(2\sin(x) - 2)(2\sin(x) - 1) = 0$, so that $\sin(x) = 1, \frac{1}{2}$. Thus $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$, are the only solutions in the specified domain.

Solution 5 I

(a) Let $\angle AFE = BFD = x$, $\angle BDF = \angle CDE = y$, $\angle CED = \angle AEF = z$. Then $\angle FAE = 180° - x - z$, $\angle FDB = 180° - x - y$, $\angle ECD = 180° - y - z$, and these three angles sum to $180°$. Therefore $540° - 2(x + y + z) = 180°$, so $x + y + z = 180°$. From ΔAEF we see that $\angle FAE + x + z = x + y + z = 180°$, so $\angle FAE = y$. Hence $\angle BDF = \angle BAC$.

(b) In a similar manner to that used in part (a) we see that $\angle ECD = \angle BFD = x$, and $\angle FBD = \angle CED = z$. Thus the triangles ABC DBF DEC and AEF are similar, and so $\frac{BD}{BF} = \frac{BA}{BC} = \frac{5}{8}$, $\frac{CD}{CE} = \frac{CA}{CB} = \frac{7}{8}$, $\frac{AE}{AF} = \frac{AB}{AC} = \frac{5}{7}$. So let $BD = 5k, BF = 8k, CD = 7l, CE = 8l, AE = 5m, AF - 7m$ for some k, l, m. Then $5k + 7l = 8$, $7m + 8k = 5$, and $5m + 8l = 7$. Solving these equations we get $k = \frac{1}{2}$, and hence $BD = \frac{5}{2}$.

SOLUTIONS to SAMPLE QUESTIONS

SENIOR - Years 11 & 12

Solution 1 S
We look first for prime factors. Since $n!$ has a factor of 13 and no factor of 17, we conclude that $13 \le n < 17$. Since 53 is a factor, so must 15 be (we only get factors of 5 from 5, 10, 15 etc.). Thus $n = 15$ or $n = 16$. Let's check the number of factors of 2 in 16! We get one factor of 2 from 2, 6, 10 and 14, we get two factors of 2 from 4 and 12 and three factors of 2 from 8 and four from 16, giving a total of 15 factors of 2, as found in $n!$ Thus $n = 16$.

Solution 2 S
For k a positive integer such that $k^2 \le n < (k+1)^2$, so that $k \le \sqrt{n} < k+1$ and so $\lfloor \sqrt{n} \rfloor = k$.

Thus for $1 \le n \le 3$, $\lfloor n \rfloor = 1$, and for $4 \le n \le 8$, $\lfloor n \rfloor = 2$, and for $9 \le n \le 15$, $\lfloor n \rfloor = 3$, etc. So the sum equals $3\times1+5\times2+7\times3+9\times4+11\times5+13\times6+2\times7=217.$

Solution 3 S
From the recurrence, we observe that there are only two choices of n which result in equations containing a_2, $n=0$ or $n=2$. These give $a_0 - a_2 = 9$, and $a_2 - 3a_0 = 25$ respectively. Adding these together gives $-2a_0 = 34$ or $a_0 = -17$.

Solution 4 S
We first calculate the side length of the equilateral triangle ABC. Let O be the centre of the smaller circle, and P the point of tangency of the circle to the side AB. Join OP and OB. Then $\angle OPB = 90°$ by tangency, and $\angle OBP = 30°$ by symmetry since $\angle CBA = 60°$. Since $OP = 1$ and ΔBOP is a right-triangle, $OB = 2$, $BP = \sqrt{3}$, and hence $AB = 2\sqrt{3}$. Also, by symmetry $CO = OB = 2$, so $CP = 3$.

Now, since $ABDE$ is a rectangle, $AE = CP = 3$, and BE is a diameter of the circumcircle. By Pythagoras, $BE^2 = AE^2 + AB^2 = 21$, so the diameter is $\sqrt{21}$.

Solution 5 S
Here is an elegant solution, using complex numbers, due to D. J. Bernstein. The circle circumscribes a square of area 5, so the circle has radius $\sqrt{\frac{5}{2}}$. Hence the rectangle has sides $\sqrt{2}$ and $\sqrt{8}$. Without loss of generality, assume that P_2P_4 has length $\sqrt{2}$. Put P_2, P_4, P_8, P_8 into the complex plane at $\sqrt{2}\left(\frac{1}{2}+i\right)$, $\sqrt{2}\left(-\frac{1}{2}+i\right)$, $\sqrt{2}\left(-\frac{1}{2}-i\right)$, $\sqrt{2}\left(\frac{1}{2}-i\right)$. Put P_1 in the complex plane at $\sqrt{\frac{5}{2}}\exp(i\theta)$; then P_3, P_5, P_7 are at $i\sqrt{\frac{5}{2}}\exp(i\theta)$, $-\sqrt{\frac{5}{2}}\exp(i\theta)$, $-i\sqrt{\frac{5}{2}}\exp(i\theta)$. The triangles P_8, P_1, P_2 and P_4, P_5, P_6 each have area $\sqrt{5}\cos(\theta)-1$. The triangles P_2, P_3, P_4 and P_6, P_7, P_8 each have area $\sqrt{\frac{5}{4}}\cos(\theta)-1$. Hence the octagon has area $3\sqrt{5}\cos(\theta)$. The maximum possible area is achieved at $\theta = 0$, and is $3\sqrt{5}$.