Mathematics Modelling Report Emily Nelson

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**⤷ How long would it take for a single E-coli bacterium to cover the Earth under ideal breeding conditions?**

***The hypothesis:***

⤷ If optimal conditions are maintained, the exponential growth of a single E. coli bacterium will allow it to cover the Earth's surface in 48 hours or less because the bacterium can reproduce rapidly under ideal circumstances, doubling approximately every 20 minutes. However, this duration may increase significantly when accounting for factors such as death rates, which can impede exponential growth and affect the overall proliferation potential of the bacteria.

***An introduction into E. coli:***

⤷ *Escherichia coli* (E. coli) is a rod-shaped bacterium, typically measuring about 1.5[µm](http://www.unicode-symbol.com/u/00B5.html) in length and 1[µm](http://www.unicode-symbol.com/u/00B5.html) in diameter. It appears as small, colourless rods under a microscope and is usually motile (capable of moving spontaneously), using flagella (microscopic hair-like structures) for movement. E. coli thrives in warm environments, particularly at around 37°C, and prefers a neutral pH (6.0 – 7.5) along with aerobic (oxygen rich) conditions. Abundant nutrients and moisture are also a necessity in order to support E. coli’s rapid reproduction.

***Report synopsis:***

⤷ This modelling report explores the hypothetical scenario of how long it would take for a single parent *Escherichia coli* (E. coli) bacterium to cover the surface of the Earth under ideal breeding conditions. We will model the growth of the bacterium, taking into account its reproduction rate, death rate, the surface area of the Earth as well as exploring factors such as exponential growth and decay and its direct correlation with the chosen topic.

***Assumptions:***

⤷ It is crucial to establish the following assumptions for the accuracy and relevance of our report. They form the foundation of our mathematical model, enabling us to effectively solve our problem.

***The primary assumptions include…***

* Initial Conditions: We start with one parent E. coli bacterium in an environment that allows the aforementioned single cell to optimally thrive.
* Reproduction Rate: Under ideal conditions, E. coli can reproduce approximately every 20 minutes (0.333 / $\frac{1}{3} $hours).
* Surface Area of the Earth: The total surface area of the Earth is approximately 510,064,471.91 million $km^{2}$. This assumes a spherical shape, as the Earth is slightly flattened (an oblate spheroid), making this calculation merely an estimate.
* Size of E. coli: An average E. coli bacterium measures about 1.5µm in length and 1µm in width. As we cannot account for every single E. coli bacterium and their unique sizes, we can instead streamline our calculations by approximating the average area of an E. coli bacterium to$1.5µm ^{2}.$
* Layering Assumption: We must assume that the E. coli bacteria stay in a single layer with no overlaps, allowing for a straightforward calculation of coverage without the complexities introduced by multiple layers.
* Earth’s properties: For this model, we must assume that the Earth's surface is composed entirely of land.

***Unpacking the problem:***

⤷ The late Garrett Hardin once summarised exponential growth with the simple statement; "what starts off slow, finishes in a flash." In this case, Hardin’s statement perfectly captures the nature of bacterial reproduction, particularly in the case of E. coli. Initially, a single bacterium's growth seems minimal and inconsequential. However, as it reproduces, the population can expand exponentially, quickly covering the Earth's surface. To successfully find solutions, we must adopt an ‘exponential’ way of thinking as opposed to a linear perspective, recognising that small beginnings can culminate in vast outcomes in a remarkably short time.

***The initial calculations:***

* The surface area of the Earth:

6,371 km

The surface area of our planet be determined using the formula $4πr^{2}$, where the Earth’s radius is 6,371 km. In order to attain a result of higher accuracy, it would be best to do the calculations by hand as opposed to using a given, rounded answer:

$$4 × π ×(6,371)^{2} $$

$$=4 × π ×40,589,641$$

$$=4 × 127, 516,117.977$$

$$≈510,064,471.91^{2}$$

$$∴The surface area of the Earth≈510,064,471.91km^{2}$$

* The size of an E. coli bacterium:

To simplify our calculations, we can treat E. coli’s typical rod shape as a rectangle. The average measurements of this bacterium are approximately 1.5µm in length with a width of 1µm. Using this simplified shape, the calculations yield an area of approximately $1.5µm ^{2} $per bacterium. This allows us to model a single layer of E. coli covering the Earth's surface.

Here are the calculations themselves:

$$1.5µm × 1µm= 1.5µm ^{2} $$

$$∴The area of an average E. coli bacterium= 1.5µm ^{2} $$

Having solved these fundamental equations, we are now able to deal with the problem itself.

* E. coli bacterium visual:

1µm

1.5µm

***Solving the problem:***

⤷ To determine how long it would take for the single E. coli bacterium to cover the surface of the Earth, we need to calculate the total number of E. coli bacteria required to achieve this coverage. Using the area of the Earth's surface and the area covered by a single bacterium, we can establish a relationship.

* Calculate the total number of E. coli required:

$$Total number of E. coli= \frac{Surface area of Earth}{Area of one E. coli}= \frac{510,064,471.91km^{2}}{1.5µm ^{2} }$$

* Converting to correct units:

However, in order to apply this formula, we must convert these numbers to the same unit of measurement. In this case, the chosen unit of measurement is $m^{2}$, meaning both the surface area of the Earth and the area of one E. coli must be converted.

**⤷** $km^{2}\rightarrow m^{2} $

$$510,064,472km^{2} ×1,000,000m^{2}/km^{2}=510,064, 472, 000,000,000m^{2}$$

$$or 5.10064472× 10^{14}m^{2}$$

**⤷** $µm ^{2}\rightarrow m^{2} $

$$1.5µm ^{2} × 10^{-12}m^{2}=0.0000000000015m^{2}$$

$$or 1.5 × 10^{-12}m^{2}$$

Having now converted the previous calculations to the correct units, we can now correctly apply the formula above:

$$Total number of E. coli= \frac{Surface area of Earth}{Area of one E. coli}= \frac{5.10064472× 10^{14}m^{2}}{1.5 × 10^{-12}m^{2}}$$

$$∴Total number of E. coli=340,042,981,333,333,333,330,000,000$$

$$or 3.40 × 10^{27} $$

At this stage, after calculating the number of E. coli required to theoretically cover the Earth, we can apply logarithmic functions to further solve our problem and determine the time needed for this growth to occur.

* $'x'$ *is the number of times the E. coli bacteria must double to successfully cover the Earth.*

$$2^{x}=340,042,981,333,333,333,330,000,000$$

$$x= log\_{2}(340,042,981,333,333,333,330,000,000)$$

$$∴x≈91.5$$

Building on this, knowing that E. coli doubles in population every $\frac{1}{3} $of an hour, we can finalise our calculations and arrive at our conclusion.

$$91.5 ÷3=30.5$$

However, 30.5 doesn’t match the 20-minute increments required for this process to occur (in this case, 0.5 is $\frac{1}{2}$ of an hour rather than being $\frac{1}{3}, \frac{2}{3}$ or $\frac{3}{3}$). To ensure that we in count full 20-minute cycles, we should round 30.5 up to 30.67. This allows us to ensure complete cell division, finalising our calculations to reach a conclusion.

Thus, it takes approximately ***30.67 hours*** or ***30 hours and 40 minutes*** for a single E. coli bacterium to multiply and cover the entire surface of the Earth.

***Extending the problem:***

⤷ Now that the initial question has been addressed, other factors that can be further explored:

* Factoring in the death rate:

According to various sources, the life expectancy of an E. coli bacterium varies a considerable amount. However, to gain a greater understanding of exponential growth and decay, we can assume that an E. coli bacterium dies every 2 hours.

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The growth and decay of *E. coli* bacteria can be easily modelled using Excel. Given that $3.40 × 10^{27} $represents the number of bacteria needed to cover the Earth's surface, we can track this value in the "population" column to determine the time required for E. coli to reach this population whilst factoring in the bacteria's death rate.

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By locating the value $3.40 × 10^{27}$ in the spreadsheet (highlighted in **pink**), it can be determined that, under the assumption that an E. coli bacterium dies every two hours, the time required for the population to reach a level sufficient to cover the Earth's surface would only an additional 40 minutes from the original answer. (A total of approximately ***31 hours and 20 minutes***).

* Speed of the E. coli bacteria:

The problem can be extended by calculating the speed at which E. coli moves over the final 20 minutes, during which the bacteria travels from halfway around the Earth to completing a full journey around its surface.

***Visual representation of the final 20 minutes (2D):***

This can be calculated using the formula…

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$$Speed= \frac{Distance}{Time}$$

The Earth's circumference is approximately 40,000 km, a value we can use to calculate the distance the *E. coli* needs to travel.

***Distance needed to travel by E. coli in final 20 minutes.***

$$40,000 ÷4=10,000 km $$

As established, E. coli doubles its population every 20 minutes. This means that in just an additional 20 minutes, or a third of an hour, the bacteria can go from covering half of the Earth's surface to engulfing it completely. This is a clear illustration of exponential growth in action.

These values can now be inserted into the formula above…

$$Speed= \frac{10,000}{\frac{1}{3}}$$

$$∴Speed=30, 000 km/h$$

It can now be determined that, in the final 20 minutes of the doubling process, the E. coli bacteria would travel at a speed of ***30,000 km/h***, which is faster than the speed of sound (343 m/s or 1,235 km/h) and even faster than commercial jet speeds, highlighting the remarkable nature of bacterial exponential growth under ideal conditions.

***Graph:***

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* Explaining the logarithmic scale:

When exponential growth is plotted on a logarithmic scale with base 10, the steep curve typical of exponential growth transforms into a straight line. This happens because the logarithmic scale compresses large values, with each unit increase on the scale representing a tenfold increase in population. Essentially, while exponential growth accelerates on a regular scale, the log scale ‘shrinks’ this acceleration, making the growth appear as a steady, linear increase. This inverse effect allows an easier method to assess the growth rate over time, even when the numbers involved are incredibly large.

***Possible further application:***

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* **Mars:** With a surface area of 144 million km², how would E. coli grow under Mars’ cold, low-pressure conditions? Adjusting the model for slower growth due to low temperatures and thin atmosphere could extend the time needed.
* **Alien Lifeforms:** Instead of Earth-based bacteria, we could hypothesise the growth of extraterrestrial organisms adapted to extreme environments, exploring their potential exponential growth on other planets.
* **Impact of Climate Change:** As climate change alters temperature and humidity, the reproduction rate of E. coli could change, affecting how quickly it spreads across different regions. Warmer temperatures might accelerate growth in some areas, while extreme heat or drought could slow it down in others.
* **Environmental Contamination:** Introducing factors like pollution or the presence of chemicals (e.g. pesticides) might inhibit bacterial growth. Modelling the impact of environmental degradation on bacterial reproduction could show how contamination affects the time to cover a given area.
* **Predation and Competition:** On Earth, bacteria face competition from other organisms and predation by viruses or larger creatures. We could factor in the impact of these biotic interactions to assess how they slow or limit the exponential growth of E. coli.

***Visualising the numbers:***

⤷ The sheer number of E. coli bacteria required to cover the Earth’s surface $(3.40 × 10^{27}) $is truly mind-boggling when compared to other cosmic scales. For instance, the Sun’s volume is so immense that about **1.3 million Earths** could fit inside it, an incredibly large number considering planetary size. If we were to equate the total number of bacteria needed to cover the Earth’s surface to the number of Earth-sized volumes that could fit inside the Sun, the number of bacteria would be approximately $3.60 × 10^{13} $**times greater** than the number of Earths that could fit inside the Sun.

We can also consider the total number of **seconds since the Big Bang**, which occurred roughly **13.8 billion years ago**. In seconds, this amounts to approximately $4.35 × 10^{17} $**seconds**. When comparing this to the number of E. coli bacteria needed to cover the Earth, we see that the bacteria outnumber the seconds since the Big Bang by **roughly**$ 10^{10} $**times**. While we often think of time since the birth of the universe as an enormous stretch of history, the number of bacteria required to cover the Earth pales in comparison to this concept of time by a staggering amount. The bacteria seem almost ‘timeless’ in their ability to proliferate at an exponential rate, surpassing not only the size of planetary volumes but also the vastness of cosmic time itself. This number is truly unfathomable.

Just imagine watching the first six seasons of *The Office*, only to return after 30 hours to a world overtaken by bacteria. The scale of their growth would be so overwhelming that you could be completely unaware of the extraordinary biological shift happening around you in such a short span of time.

***Conclusion:***

⤷ Given that the ideal conditions are provided where a single Escherichia coli (E. coli) bacterium reproduces exponentially every 20 minutes, it would take approximately ***30.67 hours*** or about ***30 hours and 40*** minutes for the population of bacteria to grow sufficiently to cover the surface area of the Earth. When factoring in the death rate of the bacteria, which extends the total time by about 40 minutes, the final estimate for the time required increases slightly to ***31 hours and 20 minutes*** *(or* ***31.33 hours***).

Furthermore, the calculated speed at which E. coli would cover the Earth's surface during the final moments of exponential growth is ***30,000 km/h***, a remarkably high speed that emphasises the rapid nature of bacterial reproduction and the power of exponential growth.

This hypothetical scenario illustrates the incredible capacity for growth in microorganisms, like E. coli under optimal conditions, as well as the limits of exponential growth in real-world contexts. Factors such as nutrient limitations, environmental conditions, and predation would, of course, drastically affect the actual potential for E. coli to cover the Earth's surface. Despite these limitations, the mathematical model offers a fascinating exploration of bacterial proliferation, leaving us with one compelling question: can this remarkable growth *truly* be achieved?

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